Analysis of Axisymmetric Shells by Transfer Matrix Method

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The transfer matrix method of analysis is convenient for one-dimensional structures. Axisymmetric shells with axisymmetric loading basically reduce to a one-dimensional problem. This method has been used to solve two typical problems. The studies depict the effectiveness and simplicity of this method for structural analysis of axisymmetric shell structures.

Nomenclature

$\{d\}^T$ = $\{uw\beta\}$ displacement vector
e = distance between the main and sub-branch center lines at the connection
I = unit matrix
N_u, N_w, N_β = force in u, w, β direction, respectively
$\{p\}^T$ = $\{N_u N_w N_\beta\}$ force vector
u, w, β = displacements in axial and perpendicular to axial direction and in rotation, respectively, in meridional plane
$[T]_{BA}$ = transfer matrix relating the state vector at section A to state vector at section B; for example, $Z_B = T_{BA} Z_A$
$\{r_d\}_{BA}$ = displacement vector due to load from section A to B
$\{r_p\}_{BA}$ = force vector due to load from section A to B
$\{Z\}^T$ = $\{dp1\}$ the extended state vector

I. Introduction

PRESSURE vessels and rocket motor casings are some of the axisymmetric shell structures of interest to aerospace engineers. For shells of revolution, under axisymmetric loading various methods of analysis such as finite difference, finite elements, with their limitations are extensively employed. More recently,[1] in the finite-element approach, efforts are made to increase accuracy by the use of curved elements, refined cone elements, numerical integration techniques, etc.

Some work on closely allied methods for associated problems now will be reviewed. Haydl[2] has extended Vlasov's method of initial functions[3] to the problem of bending of cylindrical shells of constant wall thickness and of isotropic material under discontinuous radial generalized loads alone. The differential equation reduces to that of a beam on elastic foundation with lateral loads and the general solution is well known. This analysis, as it stands, is not valid if the axial loads act and the thickness and mechanical properties vary in the direction of the axis of the shell. Goldberg et al.[4] used a numerical integration technique to solve the dynamic problem of pressurized conical shells reducing it basically to an initial value problem. This method with its limitations, as it stands, seems to be ineffective if the shells have branched parts. Further, one has to integrate numerically a large number of differential equations, and the process has its associated problems.

Transfer matrix approach[5] is very convenient for one-dimensional problems. Though this approach is used to solve beam problems extensively and in some cases plate problems, there seems to be no record of earlier work wherein the method is used to solve shell problems. The transfer matrix[6] could easily be obtained from the stiffness matrix as shown in Sec. II. Using transfer matrices, one traverses from one end to the other and solves the set of simultaneous equations.

By use of a large number of elements or better elements, one normally expects better accuracy, whether one uses the finite element or transfer matrix method of solution. This is because the transfer matrix is based on the same element stiffness matrices derived by the finite element method. However, in the transfer matrix method, one deals at the element level and the memory required is small. In the finite element method, the assembled final matrix takes much more memory and it is all the more true in the case of branched systems because of increase in bandwidth.

In this paper, the transfer matrix method is used to analyze the rocket motor case with interstage, the example given in Ref. 7. The stiffness matrix of conical element derived by numerical integration (instead of using closed-form solutions of Ref. 7) is used for deriving the transfer matrix. The effect of centerline offset on the distribution of stresses is shown and it is interesting to note that the effect can be considerable. We often encounter this problem in the design of rocket motors, pressure vessels, etc., especially when made of dissimilar or composite materials. All the earlier published methods of solution are not useful for solving the above problems.

II. Formulation and Analysis

Let $[k]$ be the stiffness matrix of conical element (Fig. 1) obtained, relating equivalent nodal forces and displacements. This matrix can be written in partitioned form. (The subscripts in the column matrices refer to the respective station values.)

$$\begin{Bmatrix} p_1 \\ \hline p_2 \end{Bmatrix} = \left[\begin{array}{c:c} k_1 & k_2 \\ \hline k_3 & k_4 \end{array}\right] \begin{Bmatrix} d_1 \\ \hline d_2 \end{Bmatrix} \tag{1}$$

Simple manipulation and rearrangement lead to

$$\begin{Bmatrix} d_2 \\ \hline p_2 \end{Bmatrix} = \left[\begin{array}{c:c} -k_2^{-1} k_1 & k_2^{-1} \\ \hline k_3 - k_4 k_2^{-1} k_1 & k_4 k_2^{-1} \end{array}\right] \begin{Bmatrix} d_1 \\ \hline p_1 \end{Bmatrix} \tag{2}$$

To complete the transformation, the sign of p_2 must be changed to account for the nature of forces, so that Eq. (2) represents a transfer matrix. Thus the transfer matrix $[T]$ is

$$[T] = \left[\begin{array}{c:c} -k_2^{-1} k_1 & k_2^{-1} \\ \hline -k_3 + k_4 k_2^{-1} k_1 & -k_4 k_2^{-1} \end{array}\right] \tag{3}$$

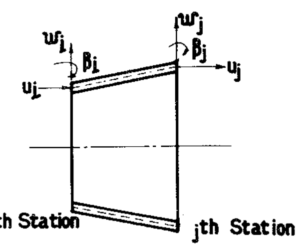

Fig. 1 An element of an axisymmetric shell.

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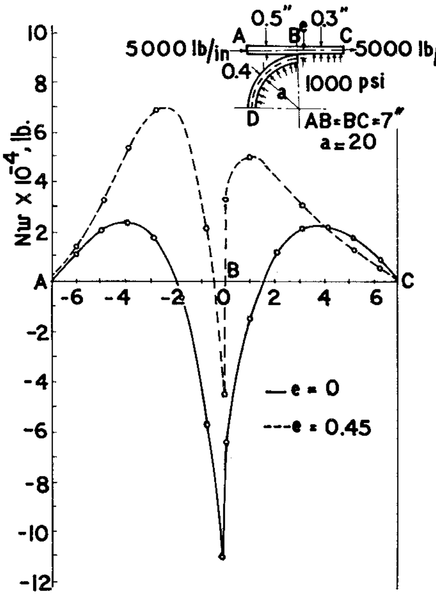

Fig. 2a Force distribution near discontinuity.

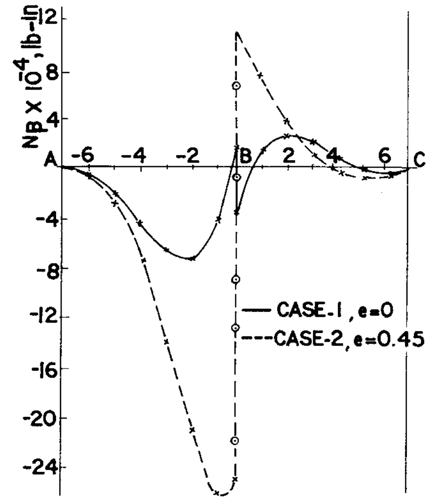

Fig. 2b Moment distribution near discontinuity.

Now, consider a shell of revolution (Fig. 2) with main branch ABC and sub-branch DB meeting at B, subjected to axisymmetric loading. Z_A, the extended state vector at A is related to $Z_B{}^L$, the state vector at left of B by the relation

$$\{Z_B{}^L\} = \begin{bmatrix} T & \vdots & r_d \\ & \vdots & r_p \\ 0 & \vdots & 1 \end{bmatrix} \{Z_A\} \tag{4}$$

At point B, generalized displacements are continuous while generalized forces are discontinuous. The effect of branch DB on bar ABC is introduced by reducing the branch to an equivalent elastic support. The state vector Z_B is related to the state vector Z_D by the relation

$$\{Z_B\} = \begin{Bmatrix} d \\ p \\ 1 \end{Bmatrix} = \begin{bmatrix} R_1 & \vdots & r_d \\ R_2 & \vdots & r_p \\ 0 & \vdots & 1 \end{bmatrix} \begin{Bmatrix} v_d \\ 1 \end{Bmatrix} \tag{5}$$

where v_d is the column vector consisting of one half number of unknown components of the state vector Z_D. R_1 and R_2 are transfer matrices which relate v_d to d and p, respectively. The elimination of v_d with simple manipulation of Eq. (5) leads to

$$p_B = R_2 R_1{}^{-1} d_B + (r_p - R_2 R_1{}^{-1} r_d) \tag{6}$$

Equation (6) gives the additional forces from branch DB contributed to bar ABC at B. The point matrix T_B relates $Z_B{}^L$ to $Z_B{}^R$, the state vector at right of B. The matrix can be written as

$$\{Z_B{}^R\} = \begin{bmatrix} I & \vdots\ 0\ \vdots & 0 \\ R_2 R_1{}^{-1} & \vdots\ I\ \vdots & r_p - R_2 R_1{}^{-1} r_d \\ 0 & \vdots\ 0\ \vdots & 1 \end{bmatrix} \{Z_B{}^L\} \tag{7}$$

Finally, the vector Z_C is expressed in terms of Z_A as

$$\{Z_c\} = \begin{bmatrix} T & \vdots & r_d \\ & \vdots & r_p \\ 0 & \vdots & 1 \end{bmatrix}_{CB} \{Z_B{}^R\}$$

$$= \begin{bmatrix} T & \vdots & r_d \\ & \vdots & r_p \\ 0 & \vdots & 1 \end{bmatrix}_{CB} [T]_B \begin{bmatrix} T & \vdots & r_d \\ & \vdots & r_p \\ 0 & \vdots & 1 \end{bmatrix}_{BA} \{Z_A\} \tag{8}$$

Equation (8) consists of six simultaneous equations in six unknowns. Once the boundary conditions are known, the state vector can be found at any intermediate section. In Eq. (8), it is assumed that the center lines of bars ABC and DB meet at a point. If ABC and DB do not meet, but are separated by a distance "e" which is generally true in case of fiber reinforced plastic rocket motor casings, then the force N_w and moment N_β contributed by the branch to B do not change. But the force N_u is transferred to ABC by equivalent force N_u and a moment $N_u e$. The point matrix T_B is suitably modified to take this factor into account.

III. Examples

A shell shown in Fig. 2 with axial loads at points A and C and pressure load in the section DBC is analyzed. The end A is allowed to deflect freely and at C the displacement u is suppressed and appropriate boundary conditions of a section of symmetry are utilized at D. Figures 2a and 2b show force and moment resultants in radial and meridional directions, respectively, for ABC with $e = 0$ and $e = 0.45$. The results for $e = 0$ agree very well with the results given in Ref. 7. It may be noted that the effect of offset e is considerable in the redistribution of force and moment distributions. Figure 3 shows displacements.

A study of stress distribution for an ellipsoidal end dome with $b/a = 0.85$ (Fig. 4) is made. From the figure, it is seen that this configuration is better than the hemispherical dome from maximum stress resultant considerations. In both examples the number of elements taken in AB, BC, and DB are 20, 40, and 40, respectively, and are equally spaced.

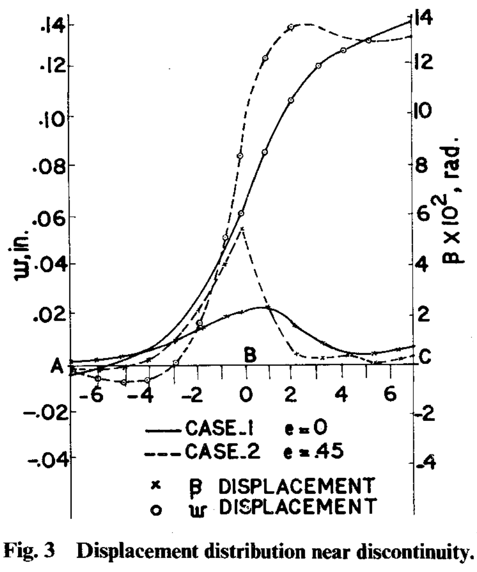

Fig. 3 Displacement distribution near discontinuity.

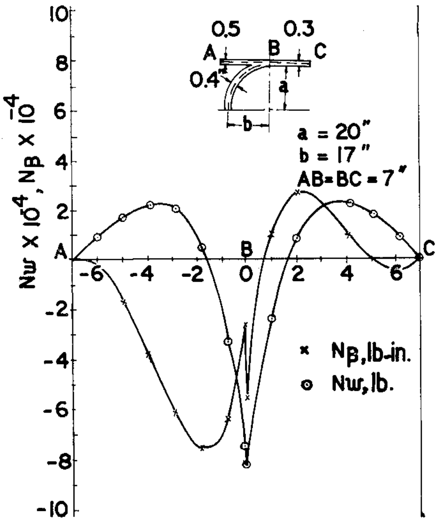

Fig. 4 Force distributions near discontinuity.

The stiffness matrix of a conical element[7] is used to derive the transfer matrix. It is observed[8,9] that for certain types of geometry and loading, the abovementioned element can give large errors even for relatively fine mesh. The stiffness matrix of a better element can definitely give better results.

IV. Conclusions and Possible Extensions

Transfer matrix method is presented for static analysis of thin shells of revolution under axisymmetric loading. Both bending and membrane forces are considered. Using an example of a shell with branches, the usefulness of the method is demonstrated especially when the centerline of main branch and sub-branches do not necessarily meet. This is common in fiber reinforced plastic structures used for pressure vessels and rocket motor cases and normally leads to secondary stresses and, in some cases if ignored, to large errors. The effect of any number of branches easily can be taken into account. There is no limit on number of elements as the memory required is modest. In contrast to this, the memory required by finite element method depends on the number of elements and bandwidth, which in turn depends on number of branches.

This method can be extended for dynamic problems, asymmetric loading, and nonlinear analysis.[7] The finite element to be chosen for asymmetric loading should include displacements in circumferential direction.

References

[1] Zienkiewicz, O. C., *The Finite Element Method in Engineering Science*, McGraw-Hill, London, 1971, p. 240.

[2] Haydl, H. A., "Bending of Cylindrical Shells by the Initial Transfer Parameter Method," *Transactions of the ASME, Journal of Engineering for Industry*, Ser. B, Vol. 95, No. 3, 1971, pp. 845–850.

[3] Vlasov, V. Z., "Method of Initial Functions-Shells," *Proceedings of the Ninth International Congress of Applied Mechanics*, Brussels, Belgium, Vol. 6, 1957, pp. 321–330.

[4] Goldberg, J. E., Bogdanoff, J. L., and Alspaugh, D. W., "Modes and Frequencies of Pressurized Conical Shells," *Journal of Aircraft*, Vol. 1, No. 6, June 1964, pp. 372–374.

[5] Pestel, E. C. and Leckie, F. A., *Matrix Methods in Elastomechanics*, McGraw-Hill, New York, 1963, pp. 153–163.

[6] Melosh, R. J., "Development of the Stiffness Method to Define Bounds on Elastic Behavior of Structures," Ph.D. thesis, 1962, Dept. of Engineering Mechanics, University of Washington, Seattle, Wash.

[7] Grafton, P. E. and Strome, D. R., "Analysis of Axisymmetrical Shells by the Direct Stiffness Method," *AIAA Journal*, Vol. 1, No. 10, Oct. 1963, pp. 2342–2347.

[8] Rashid, Y. R., "Analysis of Axisymmetric Composite Structures by the Finite Element Method," *Nuclear Engineering and Design*, Vol. 3, 1966, pp. 163–182.

[9] Jones, R. E. and Strome, D. R., "A Survey of the Analysis of Shells by the Displacement Method," proceedings of the conference *Matrix Methods in Structural Mechanics*, held at Wright-Patterson Air Force Base, Ohio, Oct. 1965, pp. 205–230, sponsored by Air Force Flight Dynamics Laboratory.